Exercise 5

Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$. Illustrate by graphing f and the tangent line.

Solution

Start by finding the corresponding y-value to $x = 0$.

$$f(0) = \sqrt{1-0} = 1$$

Then find the slope of the tangent line to the function at $x = 0$ by computing $f'(x)$,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\sqrt{1-x} \\ &= \frac{1}{2}(1-x)^{-1/2} \cdot \frac{d}{dx}(1-x) \\ &= \frac{1}{2\sqrt{1-x}} \cdot (-1) \\ &= -\frac{1}{2\sqrt{1-x}}, \end{aligned}$$

and plugging in $x = 0$.

$$f'(0) = -\frac{1}{2\sqrt{1-0}} = -\frac{1}{2}$$

Now use the point-slope formula to obtain the equation of the line going through $(0, 1)$ with slope $-1/2$.

$$y - f(0) = f'(0)(x - 0)$$

$$y - 1 = -\frac{1}{2}x$$

$$y = -\frac{1}{2}x + 1$$

Therefore, the linearization of the function $f(x)$ at $a = 0$ is

$$L(x) = -\frac{1}{2}x + 1.$$

Compare the function and its linearization for $\sqrt{0.9} = \sqrt{1-0.1}$.

$$f(0.1) = \sqrt{0.9} \approx 0.948683 \qquad L(0.1) = -\frac{1}{2}(0.1) + 1 = 0.95$$

Compare the function and its linearization for $\sqrt{0.99} = \sqrt{1-0.01}$.

$$f(0.01) = \sqrt{0.99} \approx 0.994987 \qquad L(0.01) = -\frac{1}{2}(0.01) + 1 = 0.995$$

Below is a plot of the function and the linearization at $a = 0$ versus x.